Design with Smooth Rational Geometric Spline Curves based on Fractional Decomposition

Georges-Pierre Bonneau, LIMSI-CNRS
France

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Abstract. *A well-known result of algebra is used to find a special parametric representation of rational curves. Some properties of this representation are given: derivatives of any order in any parameter value are written in a closed form, and an exact integration formula for non-parametric rational curves is given. These results are applied to find rational geometric spline curves which interpolate or approximate given points, and minimize an energy functional.*

§1. Introduction.

NURBS curves, rational Bézier curves and rational geometric spline curves are important examples of rational curves. They are often preferred to polynomial curves, because they offer more flexibility to the designer, in particular the ability to exactly describe conics [3]. Eventhough powerful CAD systems already have the ability to work with these curves, their use still raises efficiency problems, especially when evaluation of derivatives or integrals is needed. This is the case for example when searching a smooth rational curve which interpolates or approximates a set of given points: evaluation of derivatives is needed to ensure the smoothness at the joints between the segments of the curve, while evaluation of integrals is necessary to minimize the bending energy of the curve. These evaluations are straightforward for polynomial curves. But for rational curves, the usual parametric representations (NURBS, Bézier) are not well adapted. Everyone who tries to calculate the third derivative of a rational cubic Bézier curve in term of its control points- and weights will notice the complexity of the problem. Also, they are no closed formulas for the integral of rational curves in Bézier form. To evaluate such an integral, one has to use a quadrature formula, which needs many point evaluation, and only gives an approximation of the result.

In this paper, a special parametric representation of rational curves is introduced, which enables a closed form for the derivatives of any order, and an exact integration formula. In §2 we describe this representation. The evaluation of derivatives and integrals is addressed in §3. These results are used in §4 to solve the problem of smooth interpolation or approximation of given points with rational curves.

§2. Fractional Decomposition of Rational Curves.

A well-known theorem of algebra gives the decomposition of any rational function into the sum of a polynomial function and rational functions with constant numerators and simple denominators [4]. These simple denominators are powers of linear functions. The application of this decomposition to a rational curve $\boldsymbol{B}(t)$ yields the following decomposition:

$$\boldsymbol{B}(t) = \sum_{i=0}^{p} \boldsymbol{a}_i t^i + \sum_{i=1}^{d} \sum_{j=1}^{m_i} \frac{\boldsymbol{b}_{ij}}{(t-\alpha_i)^j} \tag{1}$$

We call equation (1) the **fractional representation of $\boldsymbol{B}$.**

In the following $\boldsymbol{p}$ and w respectively denote the numerator and the denominator of $\boldsymbol{B}$:

$$\boldsymbol{B}(t) = \frac{\boldsymbol{p}(t)}{w(t)}. \tag{2}$$

- The polynomial $\sum_{i=0}^{p} \boldsymbol{a}_i t^i$ is called the *polynomial part* of the representation (1). It is equal to the euclidian polynomial division of the numerator **p** by the denominator w.
- The rational function $\sum_{i=1}^{d} \sum_{j=1}^{m_i} \frac{\boldsymbol{b}_{ij}}{(t-\alpha_i)^j}$ is the *rational part* of the representation (1).
- The scalars $(\alpha_i)_{i=1,\cdots,d}$ are in general complex. They are the roots of the denominator w. If one of these roots is complex, then its conjugate is also a root.
- The integers $(m_i)_{i=1,\cdots,d}$ denote the orders of the roots $(\alpha_i)_{i=1,\cdots,d}$.
- If α_i is a real root of the denominator w, then $(\boldsymbol{b}_{ij})_{j=1,\cdots,m_i}$ are real vectors. If α_{i_0} and $\alpha_{i_1} = \overline{\alpha}_{i_0}$ are two conjugate complex roots of the denominator w, then they have the same order ($m_{i_0} = m_{i_1}$), and the corresponding vectors have complex conjugate coordinates ($\boldsymbol{b}_{i_1 j} = \overline{\boldsymbol{b}}_{i_0 j}; j = 0, \cdots, m_{i_0}$).

§3. Differentiation and Integration

The closed form of the k-th order derivative of a rational curve $\boldsymbol{B}$ in fractional representation is given by:

$$B^{(k)}(t) = \sum_{i=k}^{p} A_{i-k+1}^{k} \boldsymbol{a}_i t^{i-k} + \sum_{i=1}^{d} \sum_{j=1}^{m_i} \frac{(-1)^k A_j^k \boldsymbol{b}_{ij}}{(t-\alpha_i)^{j+k}}, \tag{3}$$

where $A_i^k = i(i+1)\cdots(i+k-1)$ are integer constants. If the order of differentiation is greater than the degree of the polynomial part ($k > p$) then this polynomial part vanishes.

The fundamental advantage between differentiation with representation (1) and differentiation with the usual parametric representation is that the number of parameters describing the curve doesn't increase with differentiation. With a Bézier representation for example, each derivation multiplies the number of parameters by two.

Furthermore, with the fractional representation (1) the parameters describing the derivatives are related to the parameters of the curve in a very simple way: the roots are unchanged, and the vectors are multiplied by an integer constant. In contrast, for Bézier curves the control-points of the derivatives are linear combinations of the control-points of the curve, and the scalar coefficients of these linear combinations are polynomial functions in the weights.

Therefore, the fractional representation is clearly better adapted for differentiation than the usual representations.

Let us know consider the integration of rational curves in fractional representation, which is straightforward: the integral function of the terms in $\frac{1}{(t-\alpha)^j}$, for $j > 1$, is $\frac{-j}{(t-\alpha)^{j-1}}$, and the integral function of $\frac{1}{t-\alpha}$ is $\ln_c(t-\alpha)$, where $z \longrightarrow \ln_c z$ is a complex logarithm function.

Therefore, we have a closed form for the integral function of a rational curve in fractional representation. This integral function can be used to compute the integral of the curve between any two parameter values.

§4. Smooth interpolation and approximation with rational curves

This section addresses the problem of finding a smooth rational curve with given geometric continuity at joints, and which interpolates or approximates given points. The classical approach is to solve a linear constraint minimization problem:

$$\begin{array}{ll} \beta \; {}^t\boldsymbol{x}E\boldsymbol{x} + (1-\beta)\; \|\boldsymbol{b} - A\boldsymbol{x}\|_2 \rightarrow min & \\ \text{subject to} \qquad B\boldsymbol{x} = \boldsymbol{d} & \end{array} \tag{4}$$

- $\boldsymbol{x}$ is the unknown which contains the vector parameters of each segment of the curve.
- $B\boldsymbol{x} = \boldsymbol{d}$ expresses the geometric continuity conditions at the joints, and also the interpolation conditions.
- $\|\boldsymbol{b} - A\boldsymbol{x}\|_2$ is equal to the squared distance between the points to approximate and the curve.
- ${}^t\boldsymbol{x}E\boldsymbol{x}$ is an energy functional.

  In this paper we choose ${}^t\boldsymbol{x}E\boldsymbol{x} = \int \|\boldsymbol{B}''(t)\|^2 dt$. This energy functional has a physical meaning: it approximates the bending energy of the curve $\boldsymbol{B}$. More information on energy functionals and their application to curve and surface smoothing is in [1,2].

- β is the smoothness parameter: $\beta = 1$ means maximum smoothing, no approximation, and $\beta = 0$ means maximum approximation.

Solving (4) means performing a blended optimization of approximation and smoothing. The novelty of our approach is to compute the approximation, interpolation and smoothing matrices A, B and E with the fractional representation as introduced in §2, and with the formulas given in §3. This increases the efficiency of the algorithm, especially for the computation of the smoothing matrix E: instead of performing a quadrature formula for each element of E, the fractional representation enables a direct evaluation of these elements.

Figures 1. to 3. show some examples.

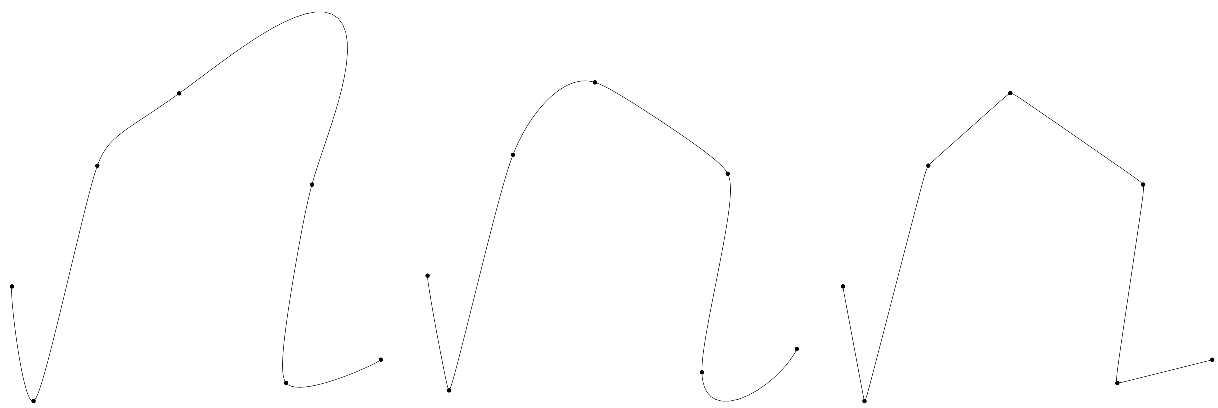

Fig. 1.: Interpolation

Figure 1. illustrates pure interpolation, without smoothing or approximation. The curves are cubic rationals with 6 segments joining with C^2-continuity. Modification of the scalar parameters (α_i) of (1) for each segment leads to a modification of the shape of the solution. If the (α_i) are chosen very far from the parameter domain, then a tension effect appears (fig. 1 right). But unlike geometric polynomial spline curves, one can obtain this tension effect while maintaining parametric continuity.

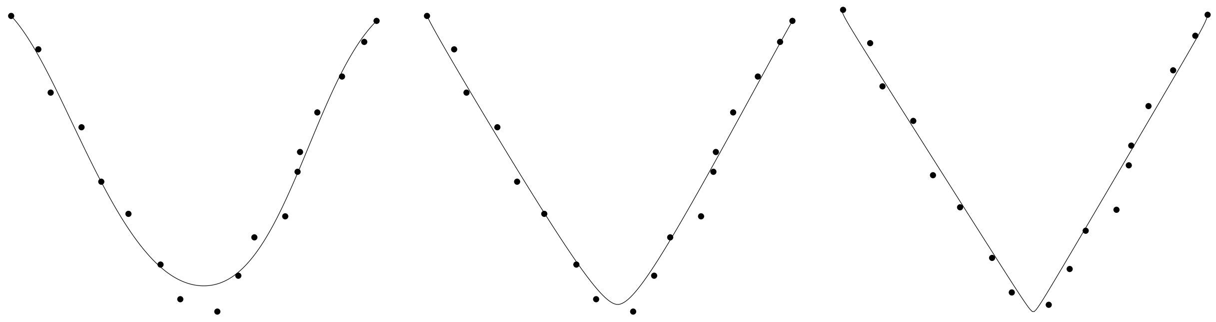

Fig. 2.: Approximation

Figure 2. illustrates the interpolation of the first and last given points and the approximation of the other points, without smoothing, by two segments of rational cubics joining with C^2-continuity. Again, by modifying the scalar parameters (α_i) of (1) for the two segments, one can control the shape of the result, and for example remove inflection points (fig. 2. middle), or create a sharp edge (fig. 2. right).

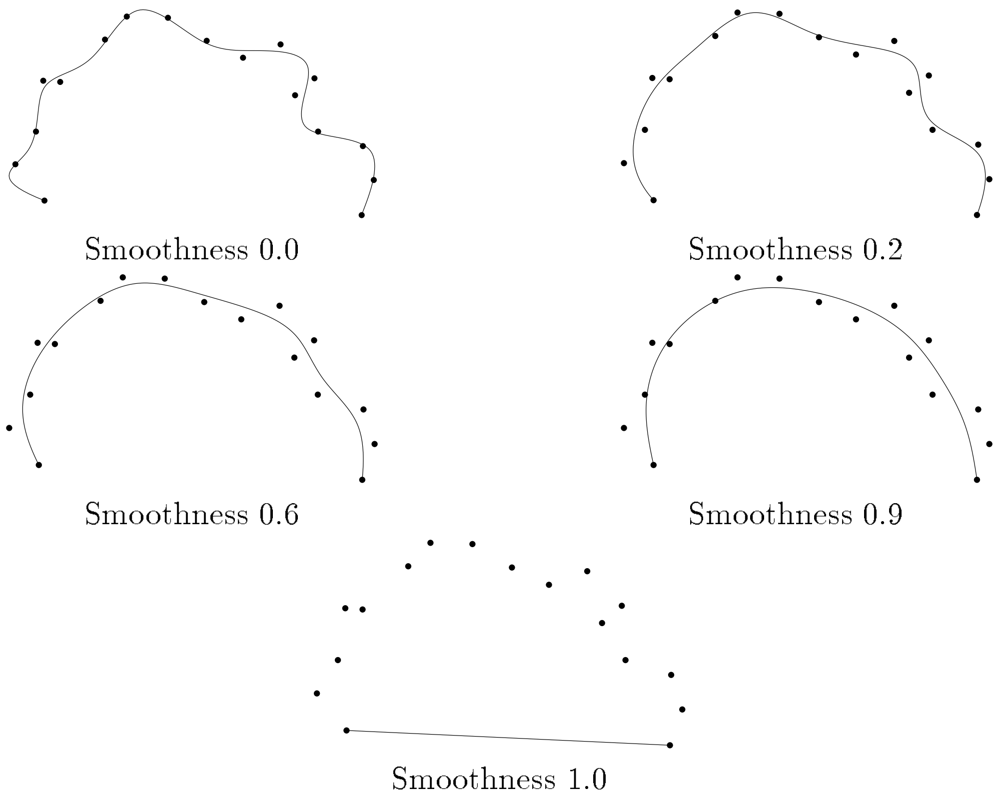

Fig. 3.: Interpolation

Figure 3. illustrates smoothing. The first and last given points are interpolated, while the other points are approximated. The curves are rational

cubics with 10 segments, joining with C^2-continuity. Increasing the smoothness parameter β in equation (4) progressively removes all inflection points. Eventually a smoothness parameter $\beta = 1$ gives the straight line between the two interpolated points.

§5. Conclusion

This paper introduced a special parametric representation of rational curves, called the fractional representation.

This representation, unlike the usual ones (Bézier, NURBS), allows a closed and simple form for the derivatives of any order, as well as an exact integral formula.

These results can be applied to any problem involving derivatives or integrals of rational curves. In this paper results concerning the smooth interpolation and approximation of given points are presented.

Research on other possible applications of the fractional representation, like degree reduction of rational curves, are currently under way.

References

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Georges-Pierre Bonneau
LIMSI-CNRS
B.P. 133
F- 91403 Orsay Cedex
France
bonneau@limsi.fr