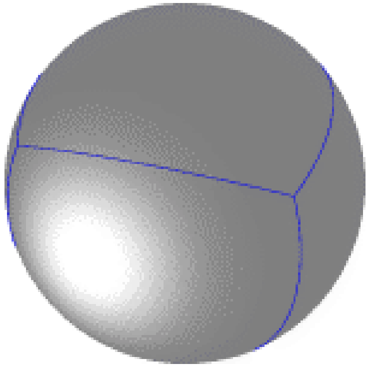
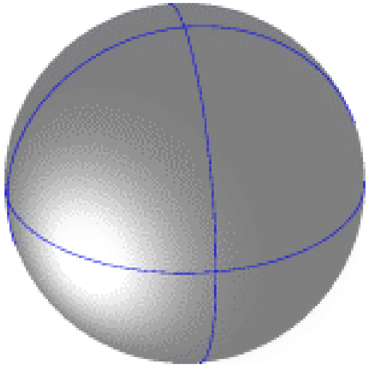
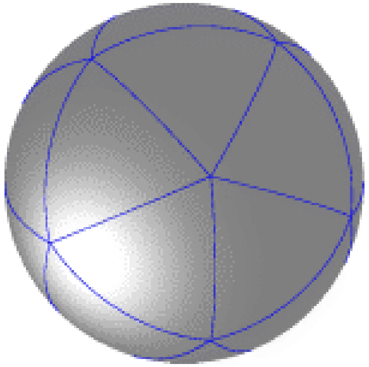
Optimal Triangular Haar Bases for Spherical Data

- Triangular Haar basis
- Orthogonality properties
- Previous bases
- New families of nearly-orthogonal bases
- Optimal basis
- Results

Triangular Haar basis: domain definition

Domain mesh = recursive 4-1 split of a base mesh

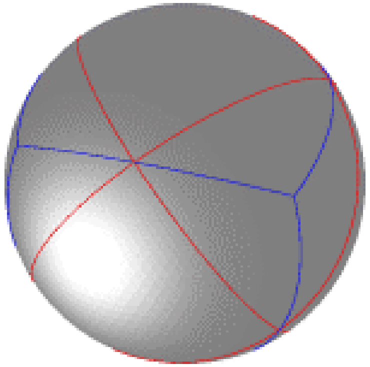
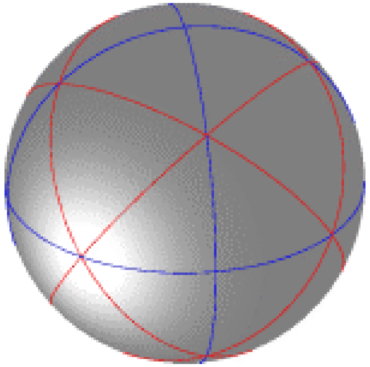
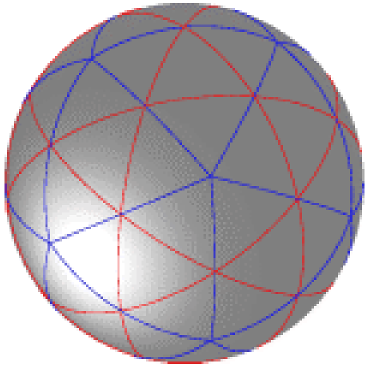
Level 0

σ (area (Δ_i))	0	0	0
	 tetraedron	 octaedron	 icosaedron

Triangular Haar basis: domain definition

Domain mesh = recursive 4-1 split of a base mesh

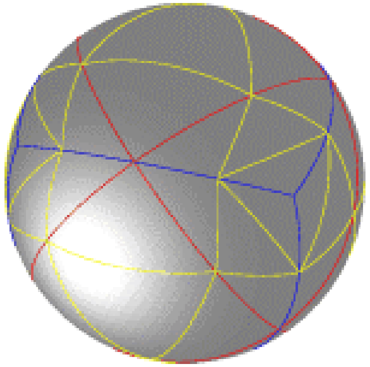
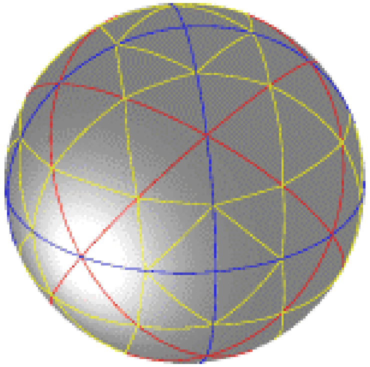
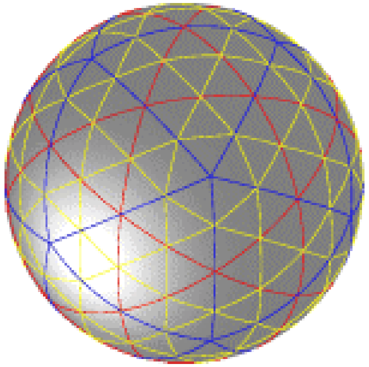
Level 1

σ (area (Δ_i))	0,1443	0,0412	0,0093
	 tetraedron	 octaedron	 icosaedron

Triangular Haar basis: domain definition

Domain mesh = recursive 4-1 split of a base mesh

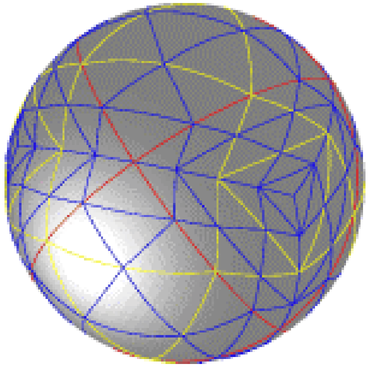
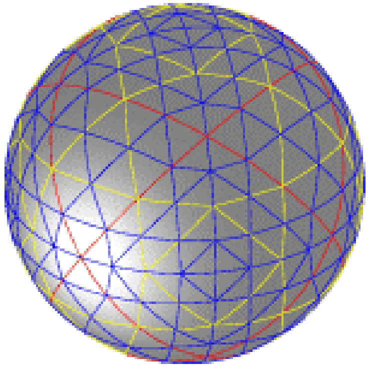
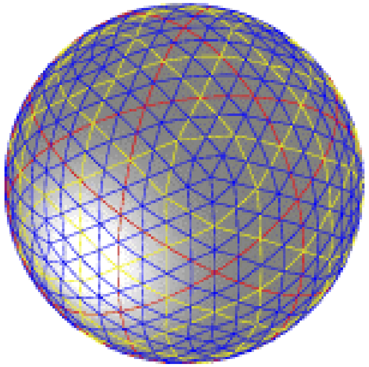
Level 2

σ (area (Δ_i))	0.0783	0.0214	0.0048
			
	tetraedron	octaedron	icosaedron

Triangular Haar basis: domain definition

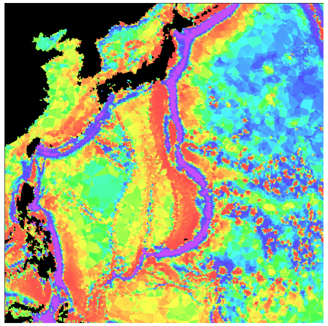
Domain mesh = recursive 4-1 split of a base mesh

Level 3

σ (area (Δ_i))	0.0394	0.0107	0.0024
	 tetraedron	 octaedron	 icosaedron

Triangular Haar Basis: Data type

Piecewise constant data: one value per face

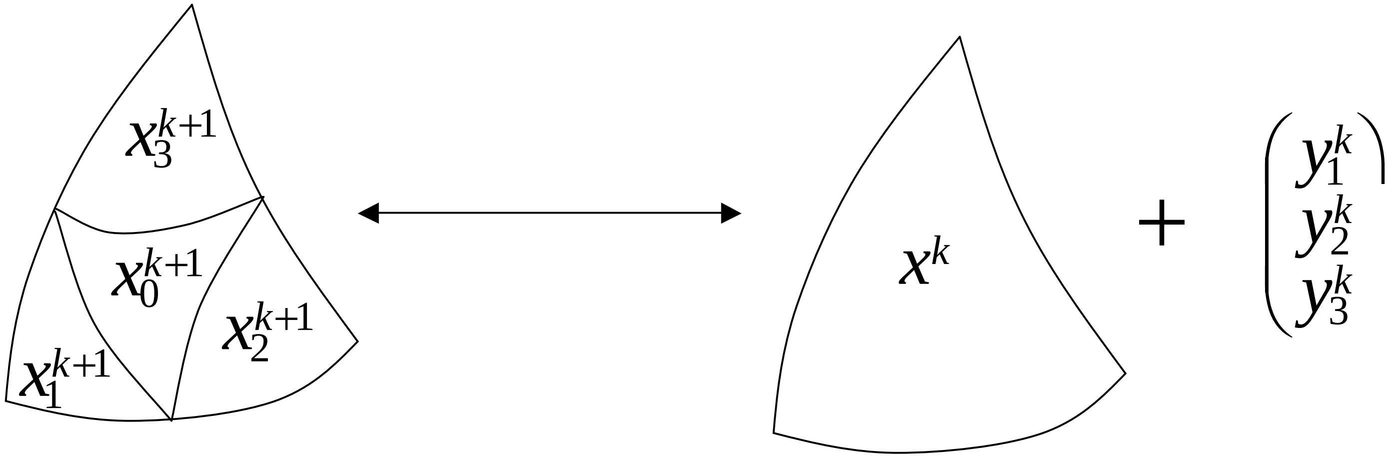


- Enough for large & complex (non-regular) data sets

- Comparison of Haar and more smoother bases in Schroeder/Sweldens SIGGRAPH'94

=> Smoother bases are not better for non-regular data sets

Analysis & Synthesis: local scheme



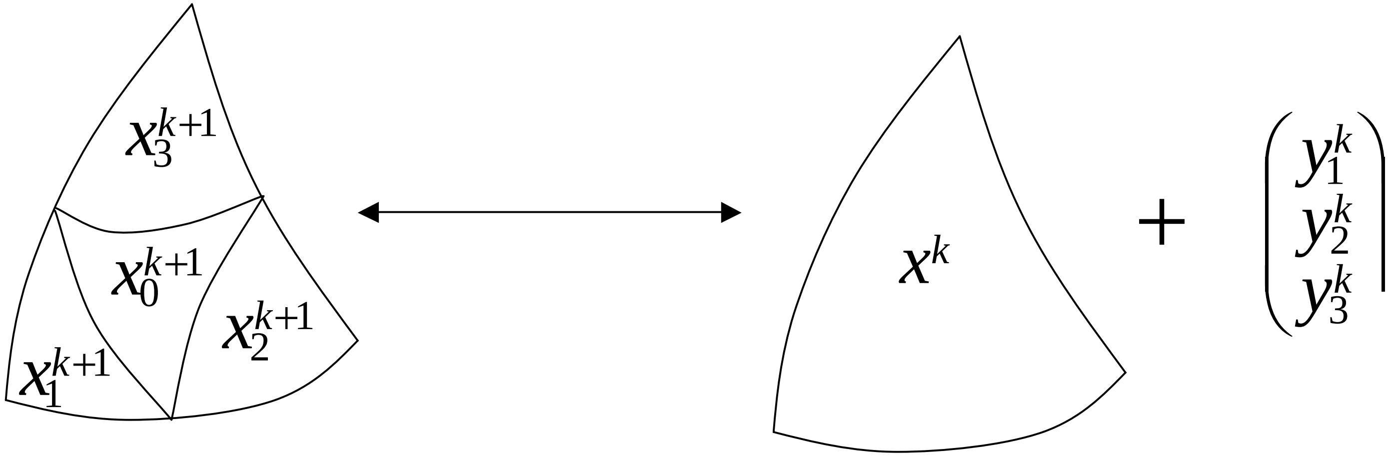
Local reconstruction

$$\begin{pmatrix} x_0^{k+1} \\ x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \end{pmatrix} = R \begin{pmatrix} x_0^k \\ y_1^k \\ y_2^k \\ y_3^k \end{pmatrix}$$

Local decomposition

$$\begin{pmatrix} x_0^k \\ y_1^k \\ y_2^k \\ y_3^k \end{pmatrix} = R^{-1} \begin{pmatrix} x_0^{k+1} \\ x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \end{pmatrix}$$

Analysis & Synthesis: functional point-of-view



$$x_0^{k+1}\Phi_0^{k+1}+x_1^{k+1}\Phi_1^{k+1}+x_2^{k+1}\Phi_2^{k+1}+x_3^{k+1}\Phi_3^{k+1}$$

$$=$$

$$x^k\Phi^k+y_1^k\Psi_1^k+y_2^k\Psi_2^k+y_3^k\Psi_3^k$$

Analysis & Synthesis: global scheme

Global decomposition (analysis)

```
for k=K-1 to 0
  for all triangles Tk at level k
    perform local decomposition in Tk // previous slide
    (xk,y1k,y2k,y3k,y4k) -> (x0k+1,x1k+1,x2k+1,x3k+1)
```

Global reconstruction (synthesis)

```
for k=0 to K-1
  for all triangles Tk at level k
    perform local reconstruction in Tk // previous slide
    (x0k+1,x1k+1,x2k+1,x3k+1) -> (xk,y1k,y2k,y3k,y4k)
```

Analysis & Synthesis: Complexity

- Hierarchy initialization:
  - Linear time required
  - Linear space required

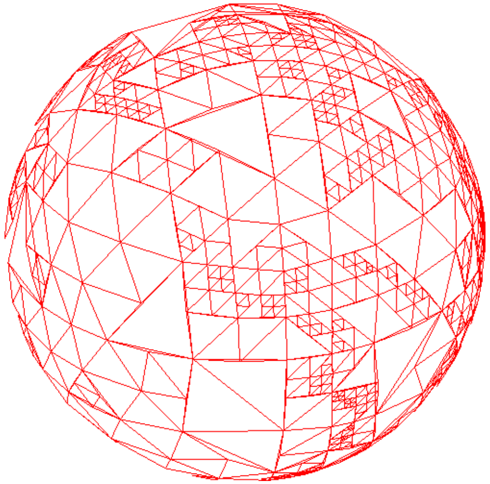
- Both global decomposition and reconstruction:
  - Linear time required
  - Linear space required

- Sort wavelet coefficients: N log N

Application: progressive transmission

Transmit & insert wavelet coefficients by decreasing order of magnitude

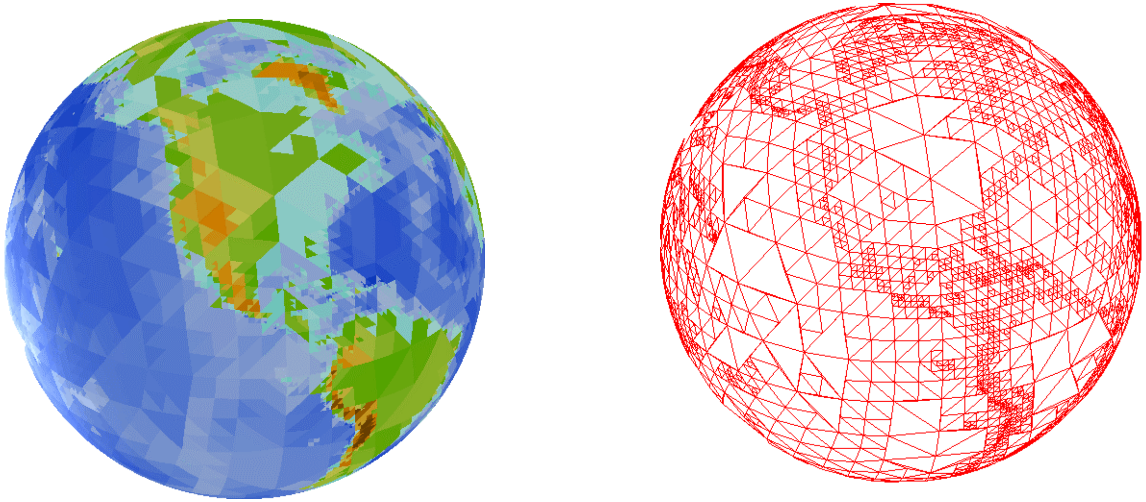
1000 wav. Coeff.



Rel. L2 error = 0.199

Application: progressive transmission

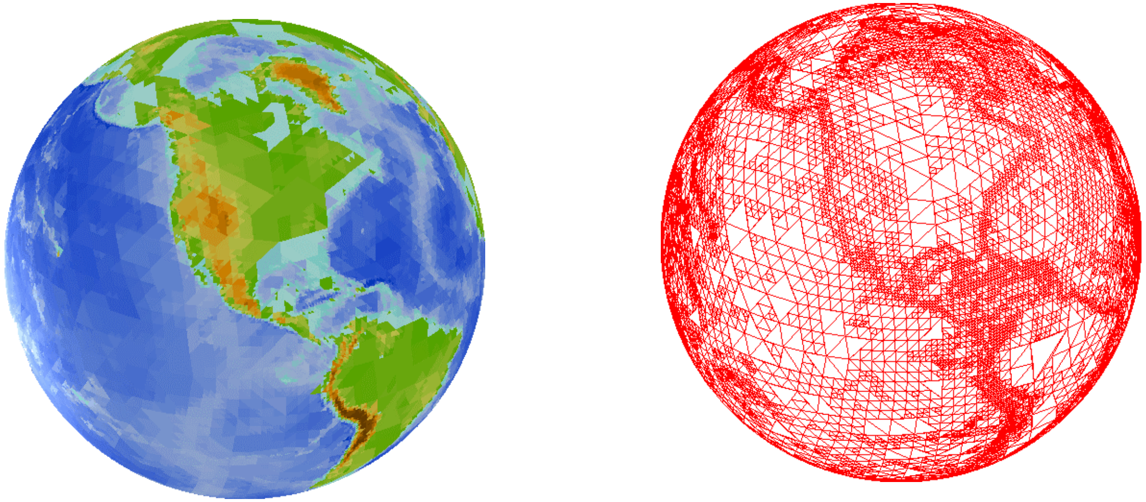
Transmit & insert wavelet coefficients by decreasing order of magnitude



5000 wav. Coeff. Rel. L2 error = 0.191

Application: progressive transmission

Transmit & insert wavelet coefficients by decreasing order of magnitude

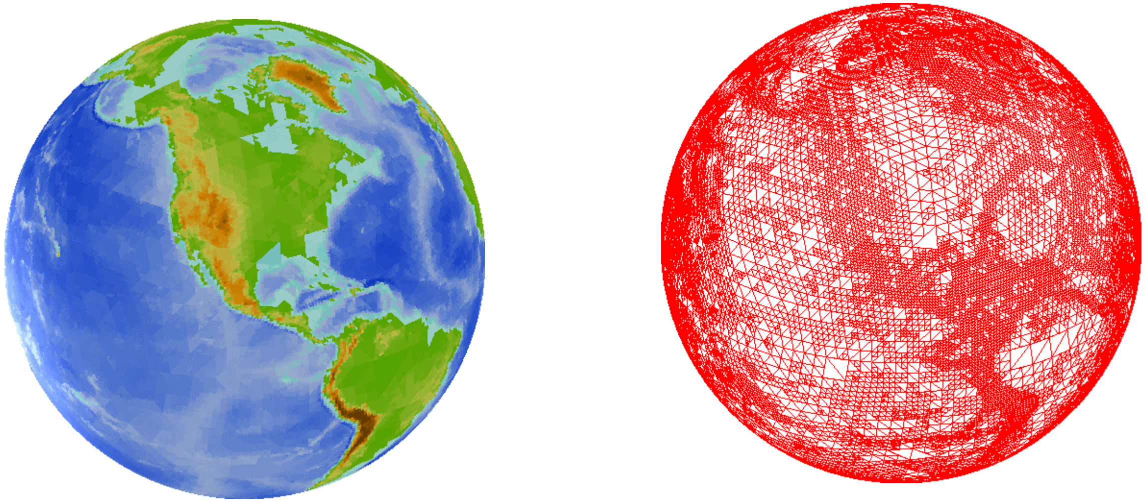


25000 wav. Coeff.

Rel. L2 error = 0.040

Application: progressive transmission

Transmit & insert wavelet coefficients by decreasing order of magnitude



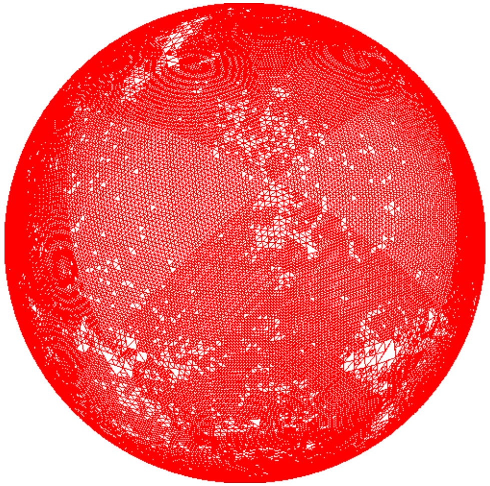
50000 wav. Coeff.

Rel. L2 error = 0.017

Application: progressive transmission

Transmit & insert wavelet coefficients by decreasing order of magnitude

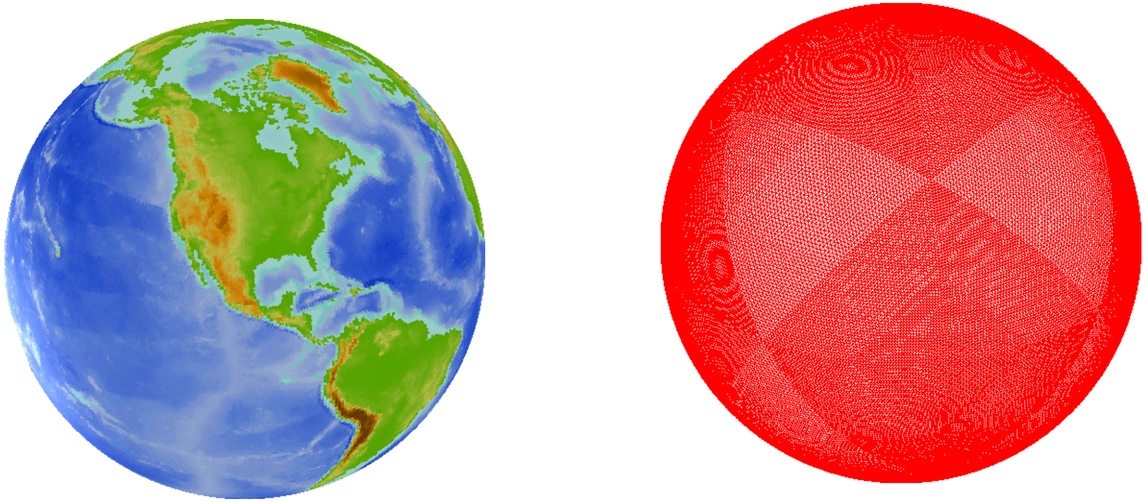
100000 wav. Coeff.



Rel. L2 error = 0.002

Application: progressive transmission

Transmit & insert wavelet coefficients by decreasing order of magnitude

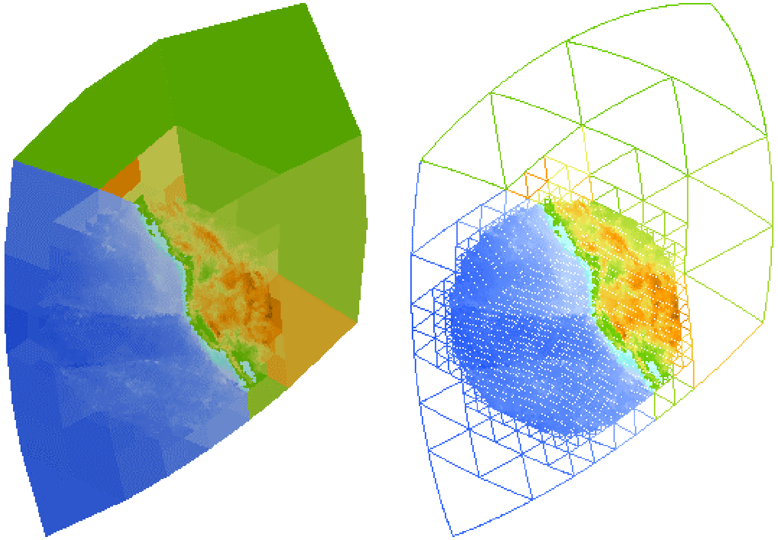


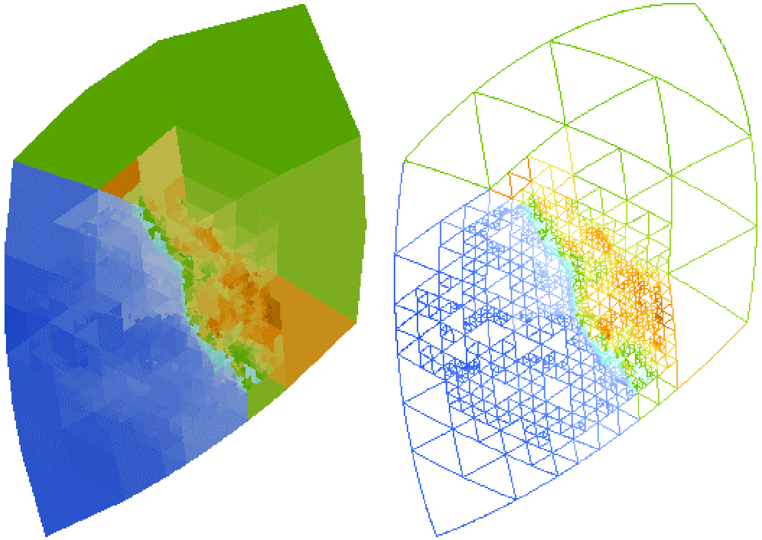
All wav. Coeff.

Rel. L2 error = 0

Application: view-dependent reconstruction

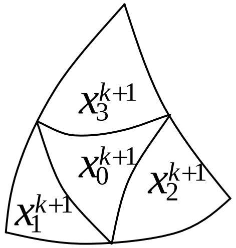
Selection of all wav. coeff.
inside a region





Selection of biggest wav. coeff.
inside a region

Orthogonality properties



$$= \quad x^k\Phi^k + y_1^k\Psi_1^k + y_2^k\Psi_2^k + y_3^k\Psi_3^k$$

- Semi-orthogonal:
  - required: $\Phi^k\Psi_1^k = \Phi^k\Psi_2^k = \Phi^k\Psi_3^k = 0$
  - prop:
    - discarding all three wavelet coefficients leads to best L2 approx.
- Orthogonal:
  - required: $\Psi_1^k\Psi_2^k = \Psi_1^k\Psi_3^k = \Psi_2^k\Psi_3^k = 0$ + semi-orthogonality
  - prop:
    - discarding the any wavelet coefficients leads to best L2 approx.
    - exact L2 error using wavelet coefficients (squared rooted sum)
    - no matrix inversion needed

Nearly orthogonality

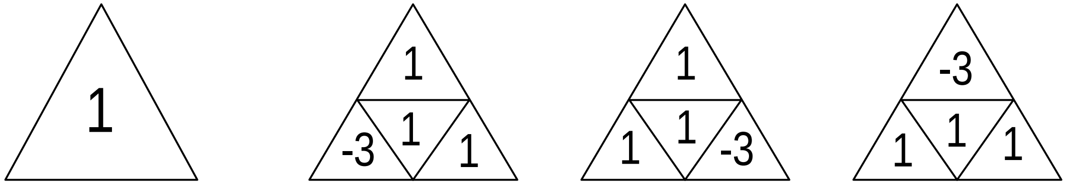
- Nielson, Vis'97
- specific to spherical wavelets:
  - no uniform triangulation of the sphere

    no orthogonal triangular spherical wavelets
- subd. depth increase => local uniform planar triang.
- Nearly orthogonality:
  - required: orthogonal in the limit case of uniform areas

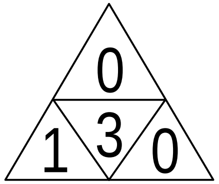
Orthogonality / Semi-orthogonality

Bio-haar basis
[Sch/Sw94],
unif. areas=1

semi-orthogonal

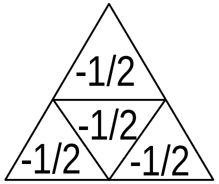


Original
function



$$=1.\Phi+\frac{1}{2}\Psi_1+\frac{3}{4}\Psi_2+\frac{3}{4}\Psi_3$$

Discarding the
lowest wav. coef.

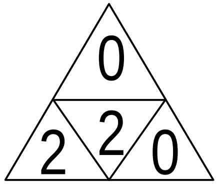


$$=1.\Phi+\frac{3}{4}\Psi_2+\frac{3}{4}\Psi_3$$

L2 error=$\sqrt{3}$

22% error loss

Best app. with
same basis
functions



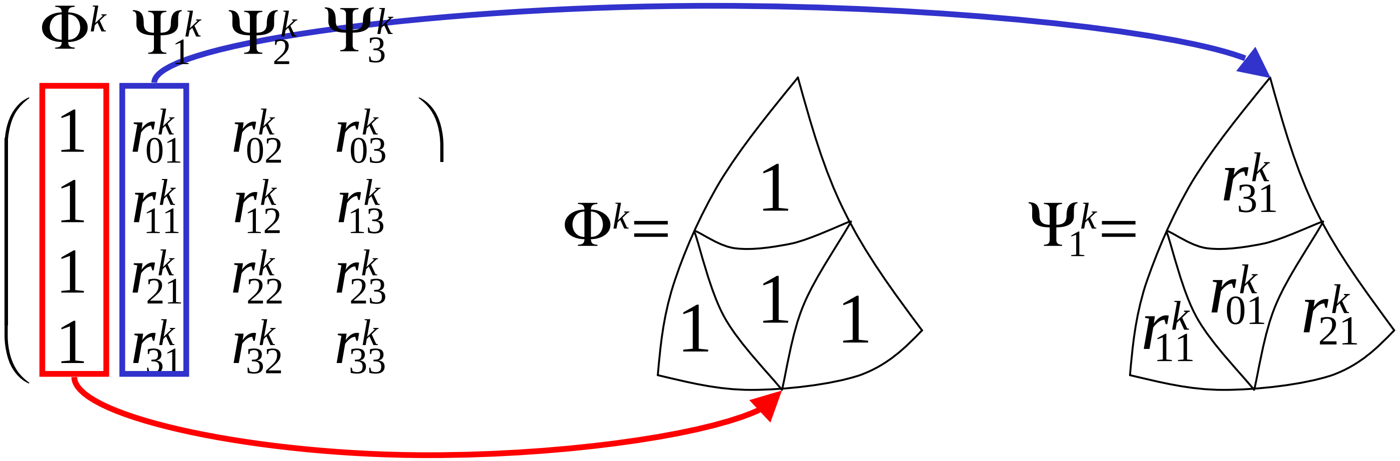
$$=1.\Phi+\frac{1}{2}\Psi_2+\frac{1}{2}\Psi_3$$

L2 error=$\sqrt{2}$

Orthogonal reconstruction matrices

Columns of 4x4 reconstruction matrix R

= value of functions $\Phi^k, \Psi_1^k, \Psi_2^k, \Psi_3^k$ on the 4 sub-triangles



$$\begin{array}{cccc} \Phi^k & \Psi_1^k & \Psi_2^k & \Psi_3^k \\ \left(\begin{array}{c} \boxed{\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}} \end{array} \right. & \boxed{\begin{array}{c} r_{01}^k \\ r_{11}^k \\ r_{21}^k \\ r_{31}^k \end{array}} & \begin{array}{c} r_{02}^k \\ r_{12}^k \\ r_{22}^k \\ r_{32}^k \end{array} & \left. \begin{array}{c} r_{03}^k \\ r_{13}^k \\ r_{23}^k \\ r_{33}^k \end{array} \right) \end{array}$$

$$\Phi^k \Psi_1^k = \alpha_0^k . \overbrace{1} \times \underbrace{r_{01}^k} + \alpha_1^k . \overbrace{1} \times \underbrace{r_{11}^k} + \alpha_2^k . \overbrace{1} \times \underbrace{r_{21}^k} + \alpha_3^k . \overbrace{1} \times \underbrace{r_{31}^k}$$

Ortho. conditions ⟷ weighted scalar product of matrix columns

Previous spherical triangular Haar bases

- Bio-Haar [Sch/Sw94]

$$\begin{pmatrix} 1 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & \alpha_0+\alpha_2+\alpha_3 & -\alpha_2 & -\alpha_3 \\ 1 & -\alpha_1 & \alpha_0+\alpha_1+\alpha_3 & -\alpha_3 \\ 1 & -\alpha_1 & -\alpha_2 & \alpha_0+\alpha_1+\alpha_2 \end{pmatrix}$$

$\alpha_0, \alpha_1, \alpha_2, \alpha_3$: triangle areas

- Semi-orthogonal: Yes

$$C1xC2 = \alpha_0 .1\times(-\alpha_1) + \alpha_1 .1\times(\alpha_0+\alpha_2+\alpha_3) + \alpha_2 .1\times(-\alpha_1) + \alpha_3 .1\times(-\alpha_1) = 0$$

$$C1xC3 = 0$$

$$C1xC4 = 0$$

- Nearly-orthogonal: No

$$\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha \qquad C2xC3 = C2xC4 = C3xC4 = -4\alpha^3 \neq 0$$

Previous spherical triangular Haar basis

- Nielson Vis'97

$$\begin{pmatrix} 1 & \Delta-\alpha_0^2-\alpha_0\alpha_1 & \Delta-\alpha_0^2-\alpha_0\alpha_2 & \Delta-\alpha_0^2-\alpha_0\alpha_3 \\ 1 & \Delta-\alpha_1^2-\alpha_0\alpha_1 & -\alpha_0\alpha_1-\alpha_1\alpha_2 & -\alpha_0\alpha_1-\alpha_1\alpha_3 \\ 1 & -\alpha_0\alpha_2-\alpha_1\alpha_2 & \Delta-\alpha_2^2-\alpha_0\alpha_2 & -\alpha_0\alpha_2-\alpha_2\alpha_3 \\ 1 & -\alpha_0\alpha_3-\alpha_1\alpha_3 & -\alpha_0\alpha_3-\alpha_2\alpha_3 & \Delta-\alpha_3^2-\alpha_0\alpha_3 \end{pmatrix}$$

where $\Delta=\alpha_0^2+\alpha_1^2+\alpha_2^2+\alpha_3^2$

$$\begin{pmatrix} 1 & \Delta-\alpha_0^2+3\alpha_0\alpha_1 & \Delta-\alpha_0^2+3\alpha_0\alpha_2 & \Delta-\alpha_0^2+3\alpha_0\alpha_3 \\ 1 & -3(\Delta-\alpha_1^2)-\alpha_0\alpha_1 & -\alpha_0\alpha_1+3\alpha_1\alpha_2 & -\alpha_0\alpha_1+3\alpha_1\alpha_3 \\ 1 & -\alpha_0\alpha_2+3\alpha_1\alpha_2 & -3(\Delta-\alpha_2^2)-\alpha_0\alpha_2 & -\alpha_0\alpha_2+3\alpha_2\alpha_3 \\ 1 & -\alpha_0\alpha_3+3\alpha_1\alpha_3 & -\alpha_0\alpha_3+3\alpha_2\alpha_3 & -3(\Delta-\alpha_3^2)-\alpha_0\alpha_3 \end{pmatrix}$$

- Matrices are both

semi- and nearly-orthogonal

New families of triangular Haar bases

- Previous reconstruction matrices:
  - degree 1 (Sch/Sw94) and 2 (Nielson97) in the triangle areas
- Systematic investigation of possible bases

  Use any polynomial function of the areas in the reconstruction matrix

- Low degree ⇒ look for degree 1 polynomial functions in $\alpha_0,\alpha_1,\alpha_2,\alpha_3$

$$\begin{pmatrix} 1 & \\ 1 & r_{ij}^{c} + \overset{3}{\underset{l=0}{\ }} r_{ij}^{l}\alpha_{l} \\ 1 & \\ 1 & \end{pmatrix}$$

4x3x5=60 free parameters

Required condition: Symmetry

- Symmetry $\Longleftrightarrow$ affine invariance
  $\Longleftrightarrow$ invariance when permuting the indices (1,2,3)

$$\begin{array}{cccccc}
 & & & 1 \longleftrightarrow & 2 & \\
 & \Bigg(& 1 & a_0+a_1\alpha_0+a_2\alpha_1+a_3\alpha_2+a_3\alpha_3 & a_0+a_1\alpha_0+a_3\alpha_1+a_2\alpha_2+a_3\alpha_3 & a_0+a_1\alpha_0+a_3\alpha_1+a_3\alpha_2+a_2\alpha_3 \\
1 \updownarrow & & 1 & b_0+b_1\alpha_0+b_2\alpha_1+b_3\alpha_2+b_3\alpha_3 & \boxed{c_0+c_1\alpha_0+c_2\alpha_1+c_3\alpha_2+c_4\alpha_3} & c_0+c_1\alpha_0+c_2\alpha_1+c_4\alpha_2+c_3\alpha_3 \\
2 & & 1 & \boxed{c_0+c_1\alpha_0+c_3\alpha_1+c_2\alpha_2+c_4\alpha_3} & b_0+b_1\alpha_0+b_3\alpha_1+b_2\alpha_2+b_3\alpha_3 & c_0+c_1\alpha_0+c_4\alpha_1+c_2\alpha_2+c_3\alpha_3 \\
 & \Bigg(& 1 & c_0+c_1\alpha_0+c_3\alpha_1+c_4\alpha_2+c_2\alpha_3 & c_0+c_1\alpha_0+c_4\alpha_1+c_3\alpha_2+c_2\alpha_3 & b_0+b_1\alpha_0+b_3\alpha_1+b_3\alpha_2+b_2\alpha_3 \Bigg)
\end{array}$$

- free parameters: $a_{0-3}, b_{0-3}, c_{0-4}$

$$\cancel{60} \xrightarrow{\qquad} \text{13 free parameters}$$

Required condition: semi-orthogonality

- C1xC2=C1xC 3=C1xC4=0

$$\begin{pmatrix} 1 & a\alpha_1+b\alpha_2+b\alpha_3 & b\alpha_1+a\alpha_2+b\alpha_3 & b\alpha_1+b\alpha_2+a\alpha_3 \\ 1 & -a\alpha_0+c\alpha_2+c\alpha_3 & -b\alpha_0-c\alpha_2 & -b\alpha_0-c\alpha_3 \\ 1 & -b\alpha_0-c\alpha_1 & -a\alpha_0+c\alpha_1+c\alpha_3 & -b\alpha_0-c\alpha_3 \\ 1 & -b\alpha_0-c\alpha_1 & -b\alpha_0-c\alpha_2 & -a\alpha_0+c\alpha_1+c\alpha_2 \end{pmatrix}$$

- free parameters: a,b,c

$\cancel{13} \longrightarrow$ 3 free parameters

Required condition: nearly-orthogonality

- C2xC3=C2xC 4=C3xC4=0 if all triangle areas are equal
- Same matrix as in previous slide, with

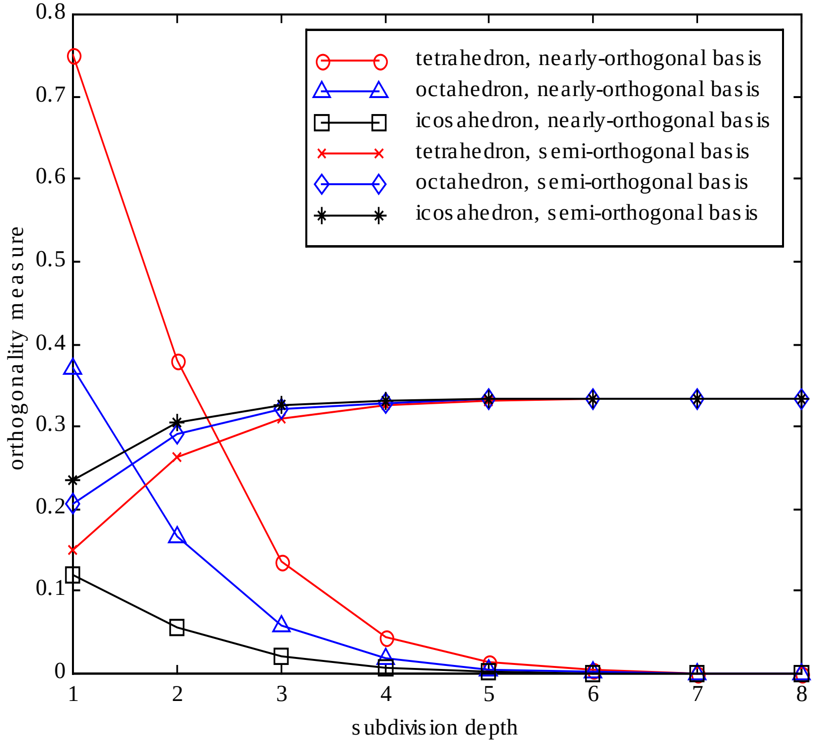
$$\left\{\begin{array}{ll} c=a+b & \longleftarrow \text{Basis family (I)} \\ \text{or} & \\ c=-\left(\frac{a+5b}{3}\right) & \longleftarrow \text{Basis family (II)} \end{array}\right.$$

- free parameters: a,b
- normalization => free parameter b, $(a=1)$

$\cancel{3} \longrightarrow$ 1 free parameter

Orthogonality measure

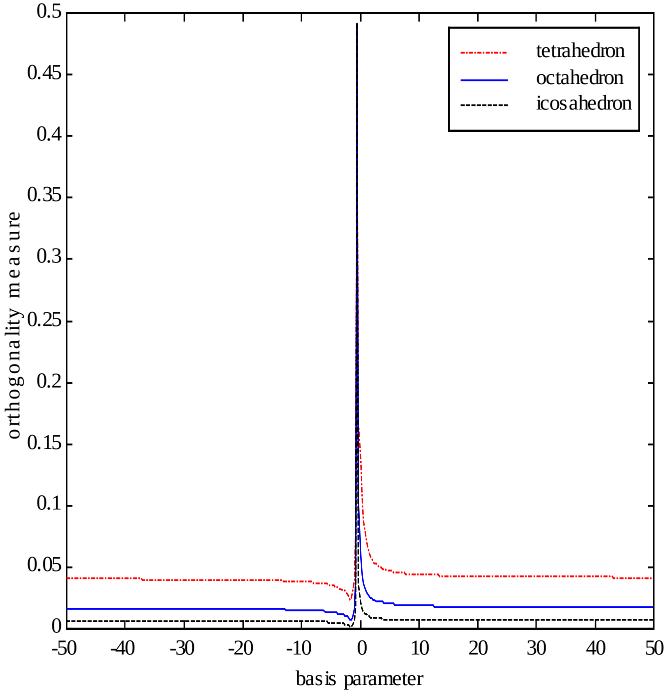
$$M(K) = \frac{1}{n} \sum_{k=1}^{K} \sum_{T^k} \left| \Psi_1^k \Psi_2^k \right| + \left| \Psi_1^k \Psi_3^k \right| + \left| \Psi_2^k \Psi_3^k \right|$$

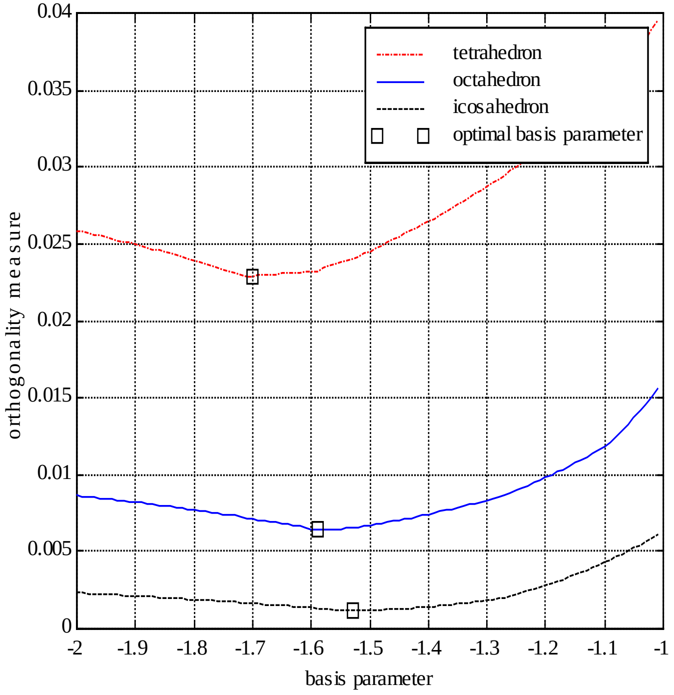


- Need to choose among all possible nearly orthogonal bases
  => optimize a measure

Basis optimization

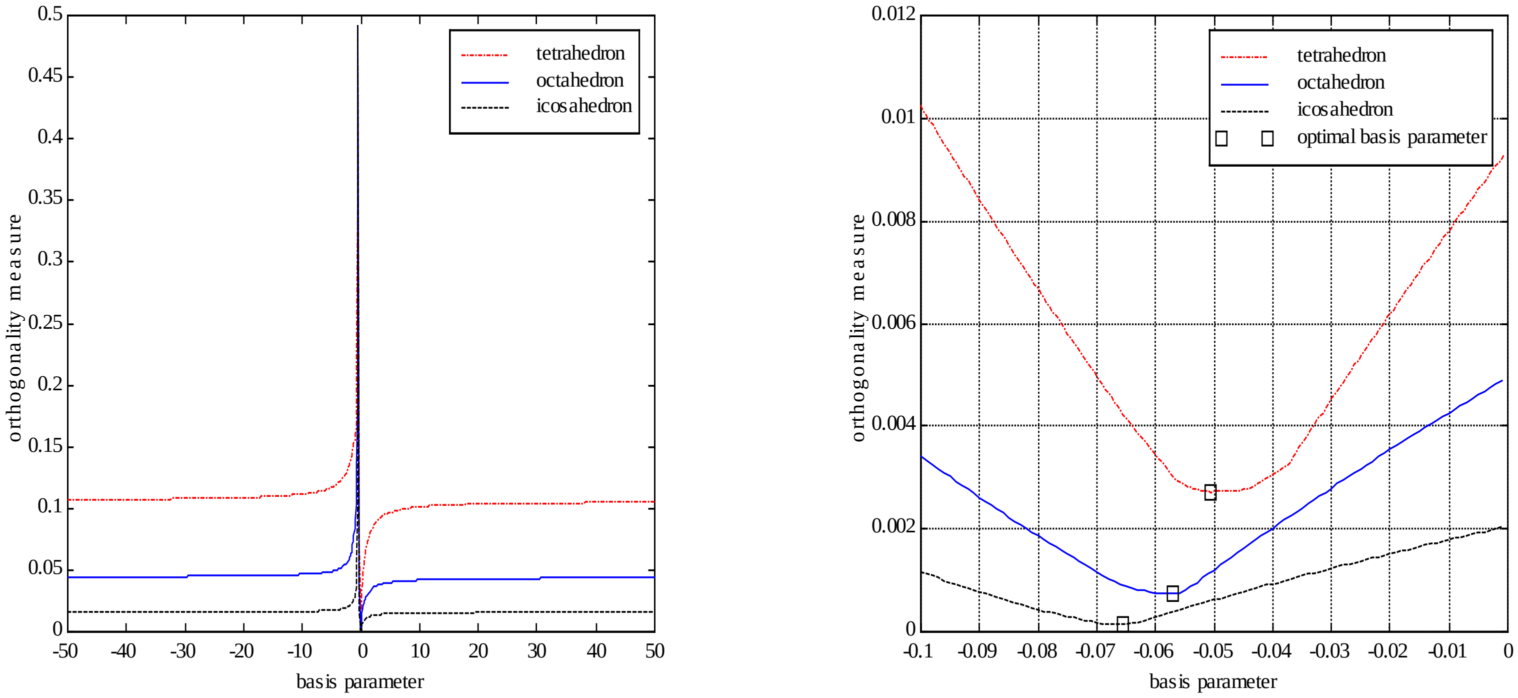
Orthogonality measure at depth 3, for different parameter values, Basis (I)





Basis optimization

Orthogonality measure at depth 3,
for different parameter values, Basis (II)



Basis optimization

		Basis (I)		Basis (II)	
Base mesh		**optimal parameter values**	**optimal orthogonality measure**	**optimal parameter values**	**optimal orthogonality measure**
tetrahedron		a=1 b=-1.707	2.28 10-2	a=1 b=-0.051	2.72 10-3
octahedron		a=1 b=-1.592	6.44 10-3	a=1 b=-0.061	8.99 10-5
icosahedron		a=1 b=-1.535	1.14 10-3	a=1 b=-0.067	1.62 10-5

Results

- Orthogonal basis => L2 error = squared rooted sum of squared discarded wavelet coefficients

- Measure relative difference between squared rooted sum of squared discarded wavelet coefficients and L2 error

  Should be as minimal as possible (vanishes exactly for orthogonal bases)

- Test data set: global topography (NOAA), mapped on a depth 7 subdivided spherical tetrahedron, octahedron & icosahedron

Results

Base mesh: spherical tetrahedron

#wav. coeff.	**comp. gain**		**Bio-Haar Sch/Sw94**	**Basis (I)**	**Basis (II)**
500	99.2%		**1.11%**	0.32%	**0.036%**
2000	96.9%		**1.76%**	0.30%	**0.034%**
6000	90.8%		**1.47%**	0.30%	**0.033%**
10000	84.7%		**1.04%**	0.30%	**0.033%**

Results

Base mesh: spherical octahedron

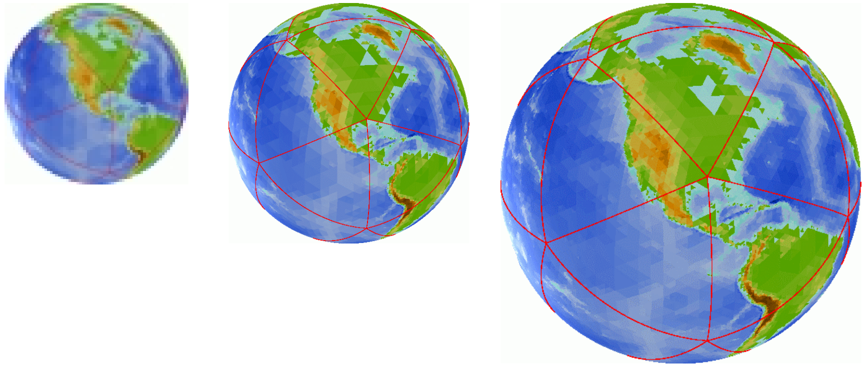
#wav. coeff.	comp. gain		Bio-Haar Sch/Sw94	Basis (I)	Basis (II)
1000	99.2%		**1.27%**	0.05%	**0.0056%**
5000	96.2%		**1.36%**	0.05%	**0.0052%**
10000	92.4%		**0.78%**	0.05%	**0.0052%**
20000	84.7%		**0.58%**	0.05%	**0.0051%**

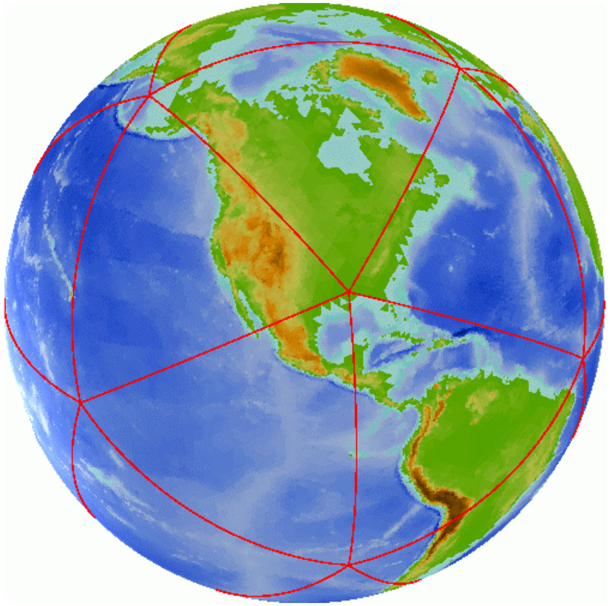
Results

Base mesh: spherical icosahedron

#wav. coeff.	**comp. gain**		**Bio-Haar Sch/Sw94**	**Basis (I)**	**Basis (II)**
5000	98.5%		**2.44%**	0.0039%	**0.00058%**
10000	96.9%		**2.30%**	0.0040%	**0.00054%**
20000	94.0%		**1.90%**	0.0040%	**0.00052%**
40000	87.8%		**1.59%**	0.0040%	**0.00051%**

Results





98.5%, 97%, 94%, 69% compression